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Some Hardy type integral inequalities

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ABSTRACT

We present new kinds of Hardy integral inequalities involving some generalization and improvement.

1. Introduction

Hardy [1] proved the following inequality. *If* $p > 1, f \geq 0$ *and*

$$F(x) = \int_0^x f(t)dt \tag{1}$$

then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{2}$$

unless $f \equiv 0$. *The constant* $(p/(p-1))^p$ *is the best possible*. Hardy's inequality plays an important role in analysis and applications.

The previous inequality still holds for parameters a and b. That is, the inequality

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx, \tag{3}$$

is valid for $0 < a < b < \infty$, see [2].

The object of this work is to give many inequalities similar to Hardy's inequality; some of them are new and others are either generalizations or improvements in some sense.

2. Results

The following result gives a generalization and improvement for Hardy's inequality in the sense when f is non-decreasing.

Theorem 2.1. *Let* $f \geq 0$, *and non-decreasing; F is as defined by* (1). *Let* $\phi \geq 0$, *and non-decreasing, and* $0 < b \leq \infty$; *then*

$$\int_0^b \phi\left(\frac{F(x)}{x}\right) dx \leq \int_0^b \phi\,(f(x))\,dx. \tag{4}$$

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In particular, by putting $\phi(x) = x^p$, $p \geq 1$, we obtain

$$\int_0^b \left(\frac{F(x)}{x}\right)^p dx \leq \int_0^b f^p(x)dx. \tag{5}$$

Proof.

$$\begin{aligned}\int_0^b \phi\left(\frac{F(x)}{x}\right) dx &= \int_0^b \phi\left(x^{-1}\int_0^x f(t)dt\right) dx \\ &\leq \int_0^b \phi\left(x^{-1}f(x)\int_0^x dt\right) dx \\ &= \int_0^b \phi\left(f(x)\right) dx. \quad \square\end{aligned}$$

The following is another generalization of Hardy's inequality.

Theorem 2.2. *$f \geq 0, g > 0, x/g(x)$ is non-increasing, $p > 1, 0 < a < 1$. Let F be as defined in (1). Then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{a(1-p)(1-a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{6}$$

In particular if we put $a = 1/p$, $g(x) = x$, we obtain Holder's inequality.

Proof.

$$\begin{aligned}\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx &= \int_0^\infty g^{-p}(x)\left(\int_0^x f(t)t^a t^{-a}dt\right)^p dx \\ &\leq \int_0^\infty g^{-p}(x)\int_0^x t^{a(p-1)}f^p(t)dt\left(\int_0^x t^{-a}dt\right)^{p-1} \\ &= \frac{1}{(1-a)^{p-1}}\int_0^\infty x^{(1-a)(p-1)}g^{-p}(x)\int_0^x t^{a(p-1)}f^p(t)dt \\ &= \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)}f^p(t)dt\int_t^\infty x^{(1-a)(p-1)}g^{-p}(x)dx \\ &\leq \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)}f^p(t)dt\left(\frac{t}{g(t)}\right)^p\int_t^\infty x^{a(1-p)-1}dx \\ &= \frac{1}{a(p-1)(1-a)^{p-1}}\int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^p dt. \quad \square\end{aligned}$$

The following result concerns the converse inequality.

Theorem 2.3. *Let $f \geq 0, g > 0, x/g(x)$ is non-decreasing, $0 < p \leq 1, a > 0$. Let F be as defined in (1). Then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geq \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{7}$$

In particular, if we put $a = 1/p$, $p > 1$, $g(x) = x$, we obtain

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geq \frac{1+p}{1-p}\left(\frac{p}{1+p}\right)^p \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{8}$$

Proof.

$$\begin{aligned}\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx &= \int_0^\infty g^{-p}(x)\left(\int_0^x f(t)t^a t^{-a}dt\right)^p dx \\ &\geq \int_0^\infty g^{-p}(x)\int_0^x t^{a(1-p)}f^p(t)dt\left(\int_0^x t^a dt\right)^{p-1} \\ &= \frac{1}{(1+a)^{p-1}}\int_0^\infty x^{(1+a)(p-1)}g^{-p}(x)\int_0^x t^{a(p-1)}f^p(t)dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{a(p-1)} f^p(t)dt \int_t^\infty x^{(1+a)(p-1)} g^{-p}(x)dx \\
&\le \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{a(p-1)} f^p(t)dt \left(\frac{t}{g(t)}\right)^p \int_t^\infty x^{a(p-1)-1}dx \\
&= \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^p dt. \quad \square
\end{aligned}
$$

A function φ is submultiplicative if $\varphi(xy) \le \varphi(x)\varphi(y)$.
The following lemma is needed for the coming result.

Lemma 2.4. *Let $\varphi \ge 0$ be convex and submultiplicative, and $\varphi(0) = 0$. Then $\varphi(x)/x$ is non-decreasing.*

Proof.

$$\left(\frac{\varphi(x)}{x}\right)' = \frac{x\varphi'(x) - \varphi(x)}{x^2} = \frac{L(x)}{x^2}.$$

$$L'(x) = x\varphi''(x) + \varphi'(x) - \varphi'(x) = x\varphi''(x) \ge 0 \quad (\varphi \text{ being convex}).$$

Then L is non-decreasing. Since $L(0) = 0$, then $L(x) \ge 0$, which implies $\left(\frac{\varphi(x)}{x}\right)' \ge 0$. Therefore $\frac{\varphi(x)}{x}$ is non-decreasing. $\square$

Theorem 2.5. *Let $\varphi \ge 0$ be convex and submultiplicative, and $\varphi(0) = 0$. Let F be as defined in (1), and let $p > 1$. Then*

$$\int_0^b x^{1-p} \frac{\varphi(F(x))}{\varphi^2(x)} dx \le \frac{1}{p-1} \int_0^b x^{2-p} \frac{\varphi(f(x))}{\varphi(x)} dx. \tag{9}$$

Proof.

$$
\begin{aligned}
\int_0^b \frac{\varphi(F(x))}{x^{p-1}\varphi^2(x)} dx &= \int_0^b \frac{1}{x^{p-1}\varphi^2(x)} \varphi\left(x\frac{F(x)}{x}\right) dx \\
&\le \int_0^b \frac{1}{x^{p-1}\varphi^2(x)} \varphi(x)\varphi\left(\frac{F(x)}{x}\right) dx \\
&= \int_0^b \frac{1}{x^{p-1}\varphi(x)} \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right) dx \\
&\le \int_0^b \frac{1}{x^{p-1}\varphi(x)} \left(\frac{1}{x}\int_0^x \varphi(f(t))\,dt\right) dx \\
&= \int_0^b \varphi(f(t)) \left(\int_t^b x^{-p} \frac{x}{\varphi(x)} dx\right) dt \\
&\le \int_0^b \varphi(f(t)) \frac{t}{\varphi(t)} \left(\int_t^b x^{-p} dx\right) dt \\
&= \frac{1}{p-1} \int_0^b t^{2-p} \frac{\varphi(f(t))}{\varphi(t)} dt. \quad \square
\end{aligned}
$$

The other type is given by the following.

Theorem 2.6. *Let $f \ge 0$, $p \ge 2$, and let F be as defined by (1). Then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \int_0^\infty x^{-1} f^{p-1}(x) F(x) dx. \tag{10}$$

Proof.

$$
\begin{aligned}
\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx &= \int_0^\infty x^{-p} F^{p-1}(x) F(x) dx \\
&= \int_0^\infty x^{-p} F^{p-1}(x) \int_0^x f(t)dtdx
\end{aligned}
$$

$$= \int_0^\infty f(t) \int_t^\infty x^{-p} F^{p-1}(x) dx dt$$
$$= \int_0^\infty f(t) \int_t^\infty x^{-p} \left(\int_0^x f(u) du \right)^{p-1} dx dt$$
$$\le \int_0^\infty f(t) \int_t^\infty x^{-p} \left(\int_0^x f^{p-1}(u) du \right) \left(\int_0^x du \right)^{p-2} dx dt$$
$$= \int_0^\infty f(t) \int_t^\infty x^{-2} \left(\int_0^x f^{p-1}(u) du \right) dx dt$$
$$= \int_0^\infty f(t) \int_t^\infty f^{p-1}(u) \int_u^\infty x^{-2} dx du dt$$
$$= \int_0^\infty f(t) \int_t^\infty f^{p-1}(u) u^{-1} du dt$$
$$= \int_0^\infty u^{-1} f^{p-1}(u) \int_0^u f(t) dt du$$
$$= \int_0^\infty u^{-1} f^{p-1}(u) F(u) du. \quad \square$$

Theorem 2.7. *Let $f \ge 0$, $\varphi \ge 0$, φ be convex, and $p > 1$. Then*

$$\int_0^\infty \varphi^p \left(\frac{F(x)}{x} \right) dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty \varphi^p \left(f(x) \right) dx. \tag{11}$$

Proof.

$$\int_0^\infty \varphi^p \left(\frac{F(x)}{x} \right) dx = \int_0^\infty \left(\varphi \left(\frac{1}{x} \int_0^x f(t) dt \right) \right)^p dx$$
$$\le \int_0^\infty \left(\frac{1}{x} \int_0^x \varphi \left(f(t) \right) dt \right)^p dx$$
$$= \int_0^\infty x^{-p} \left(\int_0^x t^{\frac{1}{p}} t^{-\frac{1}{p}} \varphi \left(f(t) \right) dt \right)^p dx$$
$$= \int_0^\infty x^{-p} \left(\int_0^x t^{1-\frac{1}{p}} \varphi^p \left(f(t) \right) dt \right) \left(\int_0^x t^{-\frac{1}{p}} dt \right)^{p-1} dx$$
$$= \left(\frac{p}{p-1} \right)^{p-1} \int_0^\infty x^{\frac{1}{p}-2} \left(\int_0^x t^{1-\frac{1}{p}} \varphi^p \left(f(t) \right) dt \right) dx$$
$$= \left(\frac{p}{p-1} \right)^{p-1} \int_0^\infty t^{1-\frac{1}{p}} \varphi^p \left(f(t) \right) \left(\int_t^\infty x^{\frac{1}{p}-2} dx \right) dt$$
$$= \left(\frac{p}{p-1} \right)^p \int_0^\infty \varphi^p \left(f(t) \right) dt. \quad \square$$

3. Other operations

Theorem 3.1. *Let $\varphi, f, g \ge 0$ be such that f, g satisfy the condition*

$$(\varphi o g) \left(\varphi' g'' + \varphi'' (g')^2 \right) \ge (1 - 1/p)(\varphi' g')^2. \tag{12}$$

Then, the following inequality holds:

$$\int_0^\infty \varphi o g \left(\frac{1}{x} \int_0^x f(t) dt \right) dx \ge \left(\frac{p}{p-1} \right)^p \int_0^\infty (\varphi o g) f(t) dt. \tag{13}$$

Proof. As by (12),

$$\begin{aligned}
\left((\varphi o g(t))^{\frac{1}{p}}\right)' &= \frac{1}{p}(\varphi o g(t))^{\frac{1}{p}-1}\varphi'(g(t))\,g'(t),\\
\left((\varphi o g(t))^{\frac{1}{p}}\right)'' &= \frac{1}{p}\left(\frac{1}{p}-1\right)(\varphi o g(t))^{\frac{1}{p}-2}\varphi'(g(t))\,g'(t)\\
&\quad+\frac{1}{p}(\varphi o g(t))^{\frac{1}{p}-1}\left(\varphi'(g(t))\,g''(t)+\varphi''(g(t))\left(g'(t)\right)^2\right)\\
&= \frac{1}{p}(\varphi o g(t))^{\frac{1}{p}-2}\left((\varphi o g(t))\left(\varphi'(g(t))\,g''(t)+\varphi''(g(t))\left(g'(t)\right)^2\right)-\left(1-\frac{1}{p}\right)\varphi'(g(t))\,g'(t)\right)\\
&\geq 0,
\end{aligned}$$

then the function $(\varphi o g(t))^{\frac{1}{p}}$ is convex. Therefore we have

$$\begin{aligned}
\int_0^\infty \varphi o g\left(\frac{1}{x}\int_0^x f(t)dt\right)dx &= \int_0^\infty\left((\varphi o g)^{\frac{1}{p}}\left(\frac{1}{x}\int_0^x f(t)dt\right)\right)^p dx\\
&\leq \int_0^\infty\left(\left(\frac{1}{x}\int_0^x (\varphi o g)^{\frac{1}{p}}f(t)dt\right)\right)^p dx\\
&= \int_0^\infty\left(\left(\frac{1}{x}\int_0^x (\varphi o g)^{\frac{1}{p}}f(t)t^{\frac{1}{p}}t^{-\frac{1}{p}}dt\right)\right)^p dx\\
&\leq \int_0^\infty\left(x^{-p}\left(\int_0^x (\varphi o g)f(t)t^{1-\frac{1}{p}}dt\right)\right)\left(\int_0^x t^{-\frac{1}{p}}\right)^{p-1}dx\\
&= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty x^{-p}\left(\int_0^x (\varphi o g)f(t)t^{1-\frac{1}{p}}dt\right)dx\\
&= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty (\varphi o g)f(t)t^{1-\frac{1}{p}}\int_t^\infty x^{-p}dxdt\\
&= \left(\frac{p}{p-1}\right)^{p}\int_0^\infty (\varphi o g)f(t)dt. \quad\square
\end{aligned}$$

We define $(\varphi g)(x)$ and $\left(\frac{\varphi}{g}\right)(x)$ by

$$(\varphi g)(x)=\varphi(x)g(x),\quad \text{and}\quad \left(\frac{\varphi}{g}\right)(x)=\frac{\varphi(x)}{g(x)},\, g(x)\neq 0.$$

Theorem 3.2. *Let $\varphi, f, g \geq 0$ be such that f, g satisfy the condition*

$$(\varphi g)\left(\varphi g''+2\varphi' g'+\varphi'' g\right)\geq(1-1/p)(\varphi g'+\varphi' g). \tag{14}$$

Then, the following inequality holds:

$$\int_0^\infty(\varphi g)\left(\frac{1}{x}\int_0^x f(t)dt\right)dx \geq \left(\frac{p}{p-1}\right)^p\int_0^\infty(\varphi g)f(t)dt. \tag{15}$$

Proof. It is not difficult to show that

$$\left((\varphi g)^{\frac{1}{p}}\right)''=\frac{1}{p}(\varphi g)^{\frac{1}{p}-2}\left((\varphi g)\left(\varphi g''+2\varphi' g'+\varphi'' g\right)-(1-1/p)(\varphi g'+\varphi' g)\right).$$

Therefore by (14), $(\varphi g)^{\frac{1}{p}}$ is convex. The rest of the proof is similar to than in Theorem 3.1. Hence it is omitted. $\square$

Theorem 3.3. *Let $\varphi, f, g \geq 0$ be such that f, g satisfy the condition*

$$\left(\frac{\varphi}{g}\right)\frac{g(\varphi'' g-\varphi g'')-2g'(\varphi' g-\varphi g')}{g^3}-(1-1/p)\left(\frac{\varphi' g-\varphi g'}{g^2}\right)\geq 0, \tag{16}$$

and $g(x) \neq 0$; then, the following inequality holds:

$$\int_0^\infty \left(\frac{\varphi}{g}\right)\left(\frac{1}{x}\int_0^x f(t)dt\right)dx \geq \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(\frac{\varphi}{g}\right) f(t)dt. \tag{17}$$

Proof. It is not difficult to show that

$$\left(\left(\frac{\varphi}{g}\right)^{\frac{1}{p}}\right)'' = \frac{1}{p}\left(\frac{\varphi}{g}\right)^{\frac{1}{p}-2}\left(\frac{\varphi}{g}\right)\frac{g(\varphi''g-\varphi g'')-2g'(\varphi'g-\varphi g')}{g^3} - (1-1/p)\left(\frac{\varphi'g-\varphi g'}{g^2}\right).$$

Therefore by (16), $(\varphi/g)^{\frac{1}{p}}$ is convex. The rest of the proof is also similar to that in Theorem 3.1 and thus it is omitted. □

Theorem 3.4. *Let $\varphi, f, \geq 0, \alpha > 0, p \geq 1$ be such that the following condition holds:*

$$(1+\alpha/p)\varphi' \geq \varphi\varphi''. \tag{18}$$

Then, the following inequality holds:

$$\int_0^\infty (\varphi^{-\alpha})\left(\frac{1}{x}\int_0^x f(t)dt\right)dx \geq \left(\frac{p}{p-1}\right)^p \int_0^\infty (\varphi^{-\alpha}) f(t)dt. \tag{19}$$

If $p \leq 1$ and (18) reverses, then (19) reverses.

Proof. It is not difficult to show that

$$\left((\varphi^{-\alpha})^{\frac{1}{p}}\right)'' = \left(\frac{-\alpha}{p}\right)\varphi^{-\frac{\alpha}{p}-2}\left(\varphi\varphi'' - (1+\alpha/p)\varphi'\right).$$

Therefore by (18), $(\varphi^{-\alpha})^{\frac{1}{p}}$ is convex. The rest of the proof is similar to than in Theorem 3.1; hence it is omitted. □

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